Mathematics

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Today we're going to cover

- Basics
- Number Theory
- Combinatorics
- Game Theory

Mathematics

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General overview

Computer Science $\subset$ Mathematics

General overview

- Usually at least one problem that involves solving mathematically.
- Problems often require mathematical analysis to be solved efficiently.
- Using a bit of math before coding can also shorten and simplify code.

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- Knowing reoccurring identities and sequences can be helpful.

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- This is called a arithmetic progression.

$$a_n = a_{n-1} + c$$

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- Depending on the situation we may want to get the n-th element

$$a_n = a_1 + (n - 1)c$$

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- Remember this one?

$$1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$$

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$$1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

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$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots$$

- More generally

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ \ldots$$

$$a_n = ar^{n-1}$$

Geometric progression

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- Or from the *m*-th element to the *n*-th

$$\sum_{i=m}^{n} ar^i = \frac{a(r^m - r^{n+1})}{(1 - r)}$$

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- And also the exponential

```
double exp(double x);
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- But the powers of 17 grow exponentially!

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- What if $k = 500$ ($\sim 1.7 \cdot 10^{615}$), or something larger?
- Impossible to work with the numbers in a normal fashion.
- Why not log?

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- But how do we do this with only ln or $\log_{10}$?
- Change base!

$$\log_b(a) = \frac{\log_d(a)}{\log_d(b)} = \frac{\ln(a)}{\ln(b)}$$

- Now we can at least count the length without converting bases

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- More generally

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

- For division

$$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$$

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- Using this identity and the ones we've covered, we get

$$x = \left\lceil (k-1) \cdot \frac{\ln(10)}{\ln(17)} \right\rceil$$

Base conversion

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- What if we actually need to use base conversion?
- Simple algorithm

```
vector<int> toBase(int base, int val) {
    vector<int> res;
    while(val) {
        res.push_back(val % base);
        val /= base;
    }
    return val;
```

- Starts from the 0-th digit, and calculates the multiple of each power.

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- What else can we do if we are working with real numbers?
- We compare them to a certain degree of precision.
- Two numbers are deemed equal of their difference is less than some small epsilon.

```
const double EPS = 1e-9;

if (abs(a - b) < EPS) {
...
}
```

Working with doubles

- Less than operator:

```
if (a < b - EPS) {
...
}
```

- Less than or equal:

```
if (a  < b + EPS) {
...
}
```

- The rest of the operators follow.

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- For example `std::set<double>`.

```
struct cmp {
    bool operator(){double a, double b}{
        return a < b - EPS;
    }
};

set<double, cmp> doubleSet();
```

- Other STL containers can be used in similar fashion.

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- This implies that we can do all the computation with integers *modulo n*.
- The integers, modulo some n form a structure called a *ring*.
- Special rules apply, also loads of interesting properties.

Modular arithmetic

Some of the allowed operations:

- Addition and subtraction modulo n

$$(a \bmod n) + (b \bmod n) = (a + b \bmod n)$$
$$(a \bmod n) - (b \bmod n) = (a - b \bmod n)$$

- Multiplication

$$(a \bmod n)(b \bmod n) = (ab \bmod n)$$

- Exponentiation

$$(a \bmod n)^b = (a^b \bmod n)$$

- *Note:* We are only working with integers.

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- We could end up with a fraction!
- Division with k equals multiplication with the *multiplicative inverse* of k.
- The *multiplicative inverse* of an integer a, is the element a^{-1} such that

$$a \cdot a^{-1} = 1 \pmod{n}$$

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  - But difficult.
  - Basis for some cryptography such as elliptic curve, Diffie-Hellmann.
- Google “Discrete Logarithm” if you want to know more.

Definitions that everybody should know

- Prime number is a positive integer greater than 1 that has no positive divisor other than 1 and itself.
- Greatest Common Divisor of two integers *a* and *b* is the largest number that divides both *a* and *b*.
- Least Common Multiple of two integers *a* and *b* is the smallest integer that both *a* and *b* divide.
- Prime factor of an positive integer is a prime number that divides it.
- Prime factorization is the decomposition of an integer into its prime factors. By the fundamental theorem of arithmetics, every integer greater than 1 has a unique prime factorization.

Extended Euclidean algorithm

– and non-extended

- The Euclidean algorithm is a recursive algorithm that computes the GCD of two numbers.

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int gcd(int a, int b){
    return b == 0 ? a : gcd(b, a % b);
}
```

- Runs in $O(\log^2 N)$.

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- Notice that this can also compute LCM

$$\text{lcm}(a, b) = \frac{a \cdot b}{\text{gcd}(a, b)}$$

- See Wikipedia to see how it works and for proofs.

Extended Euclidean algorithm

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  which simply states that there always exist x and y such that the equation above holds.
- The extended Euclidean algorithm computes the GCD and the coefficients x and y.
- Each iteration it add up how much of b we subtracted from a and vice versa.

Extended Euclidean algorithm

```
int egcd(int a, int b, int& x, int& y) {
    if (b == 0) {
        x = 1;
        y = 0;
        return a;
    } else {
        int d = egcd(b, a % b, x, y);
        x -= a / b * y;
        swap(x, y);
        return d;
    }
}
```

Applications

- Essential step in the RSA algorithm.
- Essential step in many factorization algorithms.
- Can be generalized to other algebraic structures.
- Fundamental tool for proofs in number theory.
- Many other algorithms for GCD

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- Working modulo n often requires division (multiplication by inverse).
- Given some $a \pmod{n}$, then the multiplicative inverse $a^{-1} \pmod{n}$ exists iff. a and n are coprime.
- It so happens that when we have from EGCD algorithm

$$ax + ny = \text{gcd}(a, n) = 1$$

then

$$a^{-1} \equiv x \pmod{p}$$

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- What if n is a prime number?
- Then every element has an inverse.
- And we can use Fermat's little theorem

$$a^{p-1} \equiv 1 \pmod{n}$$

- which implies that

$$a^{p-1} \cdot a^{p-2} \equiv 1 \pmod{n} \quad \Rightarrow \quad a^{-1} = a^{p-2} \pmod{n}$$

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- Using the repeated squaring method, we can compute the inverse in $O(\log N)$.
- This method only works when working modulo a **prime**.

Chinese remainder theorem

What is the lowest number *n* such that when divided by

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When stated mathematically, find *n* where

$$n \equiv 2 \pmod 3$$
$$n \equiv 3 \pmod 5$$
$$n \equiv 2 \pmod 7$$

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Let $n_1, n_2, \ldots, n_k$ be pairwise coprime positive integers, and let x be the solution to the system of linear congruences

$$\begin{aligned} x &\equiv b_1 \pmod{n_1} \\ x &\equiv b_2 \pmod{n_2} \\ &\vdots \\ x &\equiv b_k \pmod{n_k} \end{aligned}$$

Chinese remainder theorem

- The Chinese remainder theorem only states that there exists a solution and it is unique modulus the product of the moduli.
- To obtain the solution x

$$x \equiv b_1 c_1 \frac{N}{n_1} + \ldots + b_k c_k \frac{N}{n_k}$$

  where $N = n_1 n_2 \cdots n_k$.
- The coefficients c_i are determined from

$$c_i \frac{N}{n_i} \equiv \pmod{n_i}$$

  (the multiplicative inverse of $\frac{N}{n_i}$ modulus n_i)
- Use EGCD to compute c_i.

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  - $O(\sqrt{N})$
- Even better: If n is not a prime, it has a prime divisor $\leq \sqrt{n}$
  - Iterate over the prime numbers up to $\sqrt{n}$.
  - There are $\sim N/\ln(N)$ primes less N, therefore $O(\sqrt{N}/\log N)$.

Primality testing

- Trial division is a deterministic primality test.
- Many algorithms that are probabilistic or randomized.
- Fermat test; uses Fermat's little theorem.
- Probabilistic algorithms that can only prove that a number is composite such as Miller-Rabin.
- AKS primality test, the one that proved that primality testing is in *P*.

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  - $O(\sqrt{N} \log \log N)$

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11	12	13	14	15
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Primes:

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Prime sieves

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Primes:

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Prime sieves

```
vector<int> eratosthenes(int n){
    bool *isPrime = new bool[n];
    memset(isPrime, 0, sizeof isPrime);
    vector<int> primes;
    for(int i = 2; i*i <= n; ++i){
        if(isPrime[i]) {
            primes.push_back(i);
            for(int j = i; j < n; j += i)
                isPrime[j] = false;
        }
    }
}
```

Integer factorization

The fundamental theorem of arithmetic states that

- Every integer greater than 1 is a unique multiple of primes.

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We can therefore store integers as lists of their prime powers.

To factor an integer n:

- Use the sieve of Eratosthenes to generate all the primes up $\sqrt{n}$
- Iterate over all the primes generated and check if they divide n, and determine the largest power that divides n.

Integer factorization

```
map<int, int> factor(int N) {
    vector<int> primes;
    primes = eratosthenes(static_cast<int>(sqrt(N+1)))
    map<int, int> factors;
    for(int i = 0; i < primes.size(); ++i){
        int prime = primes[i], power = 0;
        while(N % prime == 0){
            power++;
            N /= prime;
        }
        if(power > 0){
            factors[prime] = power;
        }
    }
    return factors;
}
```

Integer factorization

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- The number of divisors

$$\sigma_0(n) = \prod_{i=1}^{k}(e_1 + 1)$$

- The sum of all divisors in *x*-th power

$$\sigma_m(n) = \prod_{i=1}^{k} \frac{(p^{(e_i+1)x} - 1)}{(p_i - 1)}$$

Integer factorization

- The Euler's totient function

$$\phi(n) = n \cdot \prod_{i=1}^{k} (1 - p)$$

Integer factorization

- The Euler's totient function

$$\phi(n) = n \cdot \prod_{i=1}^{k} (1 - p)$$

- Euler's theorem, if a and n are coprime

$$a^{\phi(n)} = 1 \pmod{n}$$

Fermat's theorem is a special case when n is a prime.

Mathematics

- Basics
- Number Theory
- Combinatorics
- Game Theory

Combinatorics

Combinatorics is study of countable discrete structures.

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Generic enumeration problem: We are given an infinite sequence of sets $A_1, A_2, \ldots A_n, \ldots$ which contain objects satisfying a set of properties. Determine

$$a_n := |A_n|$$

for general n.

Basic counting

- Factorial

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

- Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Basic counting

- Factorial

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- Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Number of ways to choose k objects from a set of n objects, ignoring order.

Basic counting

Properties

- $$\binom{n}{k} = \binom{n}{n-k}$$
- $$\binom{n}{0} = \binom{n}{n} = 1$$
- $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Basic counting

Pascal triangle!

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
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Basic counting

Other useful identities

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Basic counting

Other useful identities

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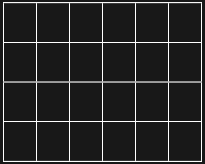
- The infamous "hockey stick sum"

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

Example

How many rectangles can be formed on a $m \times n$ grid?



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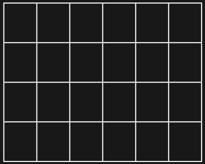

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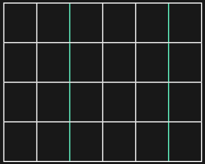

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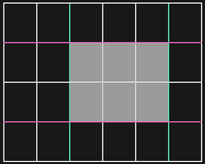

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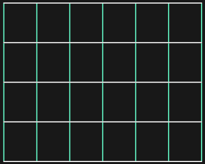

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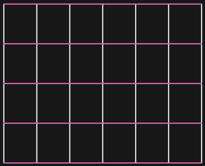

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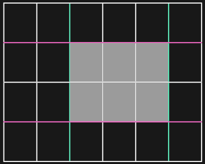

- A rectangle needs 4 edges, 2 vertical and 2 horizontal.
  - 2 vertical
  - 2 horizontal
- Total number of ways we can form a rectangle

$$\binom{n}{2}\binom{m}{2} = \frac{n!m!}{(n-2)!(m-2)!2!2!} = \frac{n(n-1)m(m-1)}{4}$$

Multinomial

What if we have many objects with the same value?

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- Number of permutations on n objects, where n_i is the number of objects with the i-th value.(Multinomial)

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- Number of way to choose k objects from a set of n objects with, where each value can be chosen more than once.

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

Example

How many different ways can we divide k identical balls into n boxes?

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- Let's imagine we have a bit string consisting only of 1 of length $n + k - 1$

$$\underbrace{1\,1\,1\,1\,1\,1\,1\ldots1}_{n+k-1}$$

Example

- Choose $n - 1$ bits to be swapped for 0

$$1 \dots 1 0 1 \dots 1 0 \dots 0 1 \dots 1$$

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$$\underbrace{1\ldots1}_{x_1}0\underbrace{1\ldots1}_{x_2}0\ldots0\underbrace{1\ldots1}_{x_n}$$

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- Number of ways to choose the bits to swap

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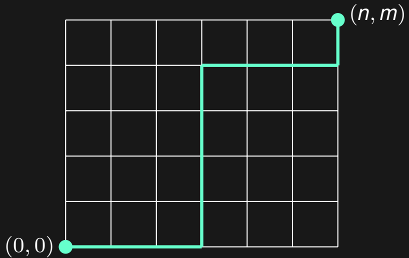
Binomial coefficients

How many different lattice paths are there from $(0, 0)$ to (n, m)?



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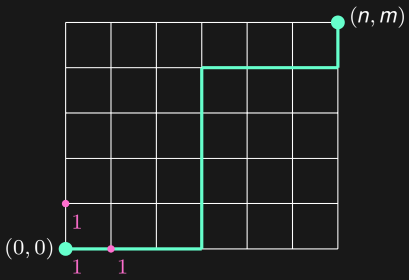

- There is 1 path to $(0, 0)$

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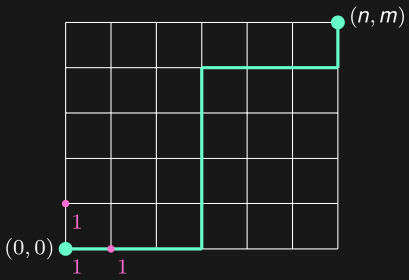

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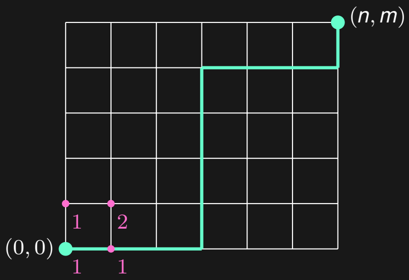

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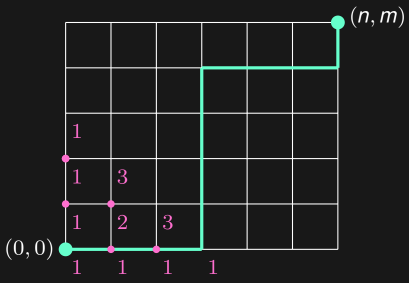

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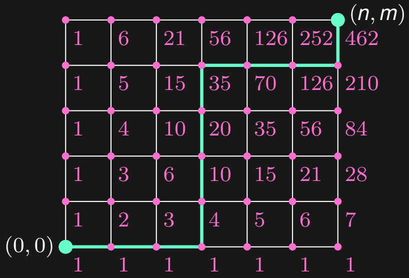

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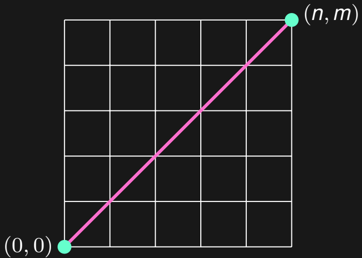
$$\binom{i+j}{i}$$

Catalan

What if we are not allowed to cross the main diagonal?



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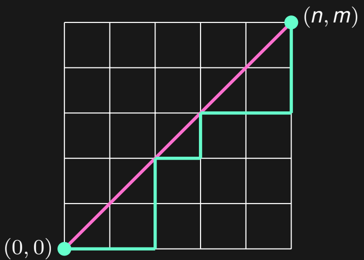

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$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

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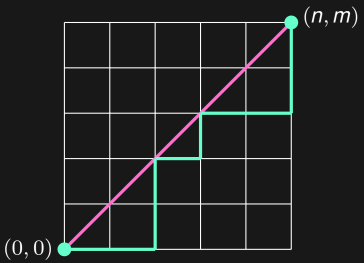

- The number of paths from $(0,0)$ to (n,m)

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- C_n are known as Catalan numbers.
- Many problems involve solutions given by the Catalan numbers.

Catalan

- Number of different ways $n + 1$ factors can be completely parenthesized.

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

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- Number of different triangulations convex polygon with $n+2$ sides

- Number of full binary trees with $n+1$ leaves.
- And aloot more.

Fibonacci

The Fibonacci sequence is defined recursively as

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

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Already covered how to calculate f_n in $O(N)$ time with dynamic programming.

Fibonacci

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$$
\begin{aligned}
f_1 &= 1 \\
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f_n &= f_{n-1} + f_{n-2}
\end{aligned}
$$

Already covered how to calculate f_n in $O(N)$ time with dynamic programming.
But we can do even better.

Fibonacci as matrix

The Fibonacci sequence can be represented by a vectors

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

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Using fast exponentiaton, we can calculate f_n in $O(\log N)$ time.

Fibonacci as matrix

Any linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \dots c_k a_{n-k}$$

can be expressed in the same way

$$\begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_{n-k} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_k \\ 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-k-1} \end{pmatrix}$$

Fibonacci as matrix

Any linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \dots c_k a_{n-k}$$

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$$\begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_{n-k} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_k \\ 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-k-1} \end{pmatrix}$$

With a recurrence relation defined as a linear function of the k preceding terms the running time will be $O(k^3 \log N)$.

Mathematics

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Game theory

Game theory is the study of strategic decision making, but in competitive programming we are mostly interested in combinatorial games.

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Example:

- There is a pile of *k* matches.
- Player can remove 1, 2 or 3 from the pile and alternate on moves.
- The player who removes the last match wins.
- There are two players, and the first player starts.
- Assuming that both players play perfectly, who wins?

Game theory

We can analyse these types of games with *backward induction*.

Game theory

We can analyse these types of games with *backward induction*.
We call a state *N*-position if it is a winning state for the next player to move, and *P*-position if it is a winning position for the previous player.

- All terminal positions are *P*-positions.
- If every reachable state from the current one is a *N*-position then the current state is a *P*-position.
- If at least one *P*-position can be reached from the current one, then the current state is a *N*-position.
- A position is a *P*-position if all reachable states form the current one are *N* position.

Game theory

Let's analyse our previous game.

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- The terminal position is a P-position.

0	1	2	3	4	5	6	7	8	9	10	11	12	...
P													

Game theory

Let's analyse our previous game.

- The terminal position is a *P*-position.
- The positions reachable from the terminal positions are *N*-positions.

0	1	2	3	4	5	6	7	8	9	10	11	12	...
P	*N*	*N*	*N*										

Game theory

Let's analyse our previous game.

- The terminal position is a *P*-position.
- The positions reachable from the terminal positions are *N*-positions.
- Position 4 can only reach *N*-positions, therefore a *P* position.

0	1	2	3	4	5	6	7	8	9	10	11	12	$\dots$
P	*N*	*N*	*N*	*P*									

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- The next 3 positions can reach the *P*-position 4, therefore they are *N*-positions.

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P	*N*	*N*	*N*	*P*	*N*	*N*	*N*						

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- The next 3 positions can reach the *P*-position 4, therefore they are *N*-positions.
- And so on.

0	1	2	3	4	5	6	7	8	9	10	11	12	...
P	*N*	*N*	*N*	*P*	*N*	*N*	*N*	*P*	*N*	*N*	*N*	*P*	...

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We can see a clear pattern of the *N* and *P* positions in the previous game. – Easy to prove that a position is *P* if $x \equiv 0 \pmod{p}$.

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- Not only one dimensional games.
- What if there are n piles instead of 1?
- What if we can remove 1, 3 or 4?

The game called Nim

- There are n piles, each containing x_i chips.
- Player can remove from exactly one pile, and can remove any number of chips.
- The player who removes the last match wins.
- There are two players, and the first player starts and they alternate on moves.
- Assuming that both players play perfectly, who wins?

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This theorem extends to other sums of combinatorial games!

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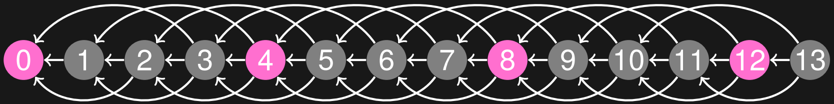
Like the first subtraction game.



We often denote the set of states(vertices) as X instead of V and edges as F instead of E.

Sprague-Grundy

Games can also be analysed with the *Sprague-Grundy* function.

- The *Sprague-Grundy* function of a graph (X, F), is a function g defined on X and taking non-negative integer values such that

$$g(x) = \min\{n \geq : \; n \neq g(y)\}$$

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- The smallest non-negative integer among the Sprague-Grundy values of the followers of x (states which x has an edge to).

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- The minimal excludant of $\{0, 1, 2, 5, 6\}$ is 3.
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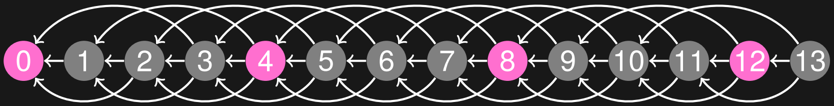
We can redefine the Sprague-Grundy function

$$g(x) = \text{mex}\{g(y) \; : \; y \in F(x)\}$$

Sprague-Grundy

The graph of the previous game.



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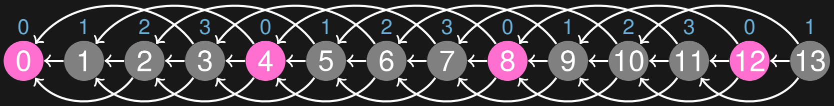
Adding the Sprague-Grundy value.



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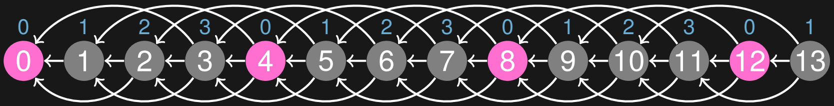

Position x is a P position iff. $g(x) = 0$.

Sum of combinatorial games

Back to Nim

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If g_i is the Sprague-Grundy function of G_i, $i = 1, 2, \ldots, n$, then $G = G_1 + G_2 + \ldots + G_n$ has the Sprague-Grundy function

$$g(x_1, x_2, \ldots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \ldots \oplus g_n(x_n)$$

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The sum of games can simply be thought of as the cartesian product of the positions, but each move consists of a move in one game. Just like Nim, where we can only remove chips from one pile.

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Take the Game Theory course if you want to know more.