Algorithmic Coincidence Classification of Mesh Patterns

Permutation Patterns 2016

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Coincidence

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- Note that for classical patterns $p \asymp p'$ if and only if $p = p'$

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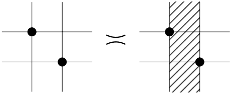

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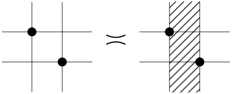
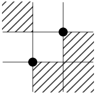
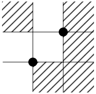


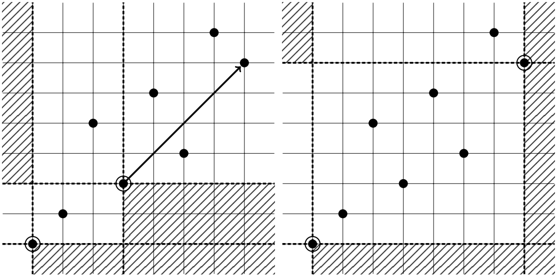
- Mesh patterns are not coincident if they have different underlying classical patterns

Previous work

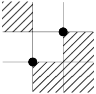
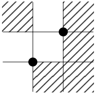
- The shading lemma [2], and the simultaneous shading lemma [1] give sufficient conditions that imply coincidence of patterns
- They are close to capturing all coincidences of mesh patterns of length $\leqslant 2$
- The idea behind the lemmas is swapping out points in an occurrence of a pattern, obtaining the objective pattern

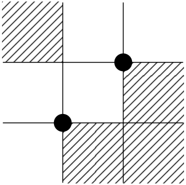
Shading lemma: Example

$p =$  $\asymp$  $= p'$



Same example: Different argument

$p =$  $\asymp$  $= p'$



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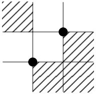
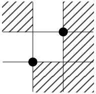
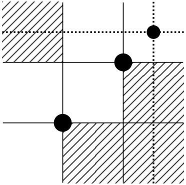
$p =$  $\asymp$  $= p'$



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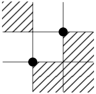
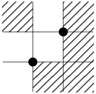
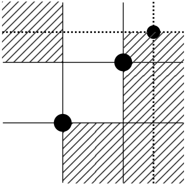
$p =$  $\asymp$  $= p'$



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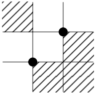
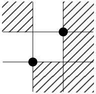
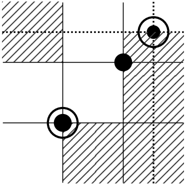
$p =$  $\asymp$  $= p'$



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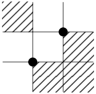
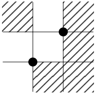
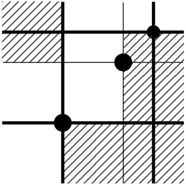
$p =$  $\asymp$  $= p'$



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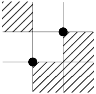
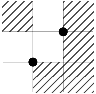
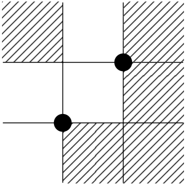
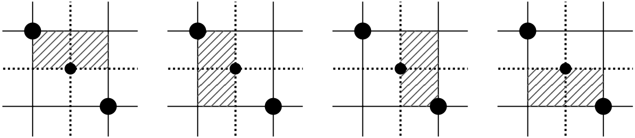
$p =$  $\asymp$  $= p'$



Formalization

Let $p = 21$. The set $p^{\ulcorner 1,1 \urcorner \star}$ consists of the mesh patterns:



$p^{\ulcorner 1,1 \urcorner \uparrow}$ $\quad$ $p^{\overleftarrow{\ulcorner 1,1 \urcorner}}$ $\quad$ $p^{\overrightarrow{\ulcorner 1,1 \urcorner}}$ $\quad$ $p^{\ulcorner 1,1 \urcorner \downarrow}$

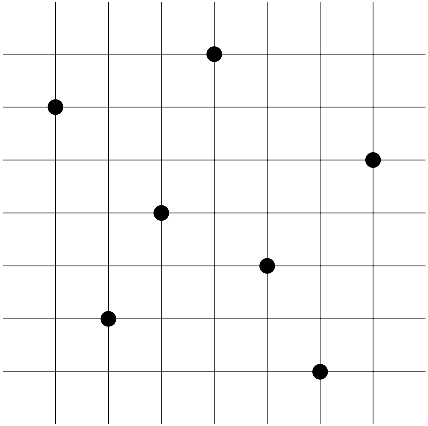
Lemma 2

Let $p = (\tau, R)$ be a mesh pattern with $\ulcorner i, j \urcorner \notin R$. If any mesh pattern in the set $p^{\ulcorner i,j \urcorner \star}$ contains a non-trivial occurrence of p then $p \asymp (\tau, R \cup \{\ulcorner i, j \urcorner\})$.

The shading lemma follows from this lemma.

Comparing occurrences

Let $p = 2413$.



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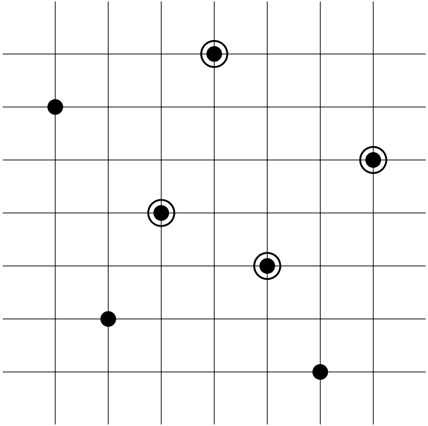

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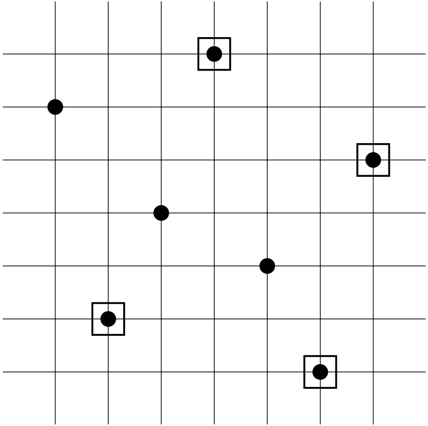

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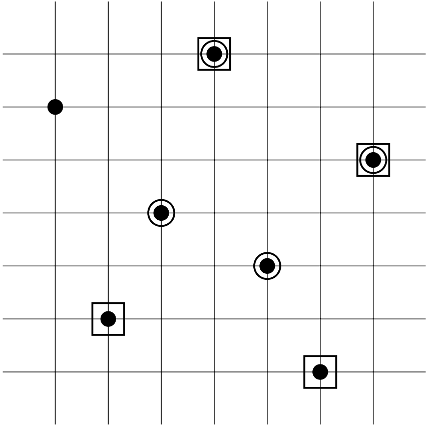

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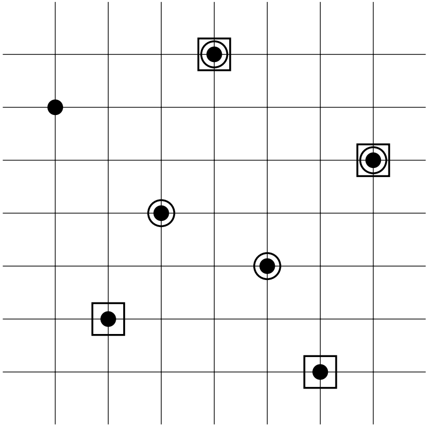


Force $F = ((4, N), (2, W), (1, W))$. The second (boxed) occurrence is stronger w.r.t. F.

Shading multiple boxes

Lemma 5

Let $p = (\tau, R)$ be a mesh pattern with force F, and $p' = (\tau, R')$ be another mesh pattern. Let $S = \{s_1, s_2, \dots, s_k\}$ where $S = R' \backslash R$. If all the sets

$$(\tau, R)^{s_1\star}, (\tau, R \cup \{s_1\})^{s_2\star}, \dots, (\tau, R \cup \{s_1, s_2, \dots, s_{k-1}\})^{s_k\star}$$

contain a non-trivial occurrence of p that is stronger than the trivial occurrence; or an occurrence of p', then containment of p implies containment of p'.

The simultaneous shading lemma follows from this lemma.

Results

Pattern	SL	Lemma 2	SSL	Lemma 5
12	237	237	229	221
123	34626	34618	34154	33634
132	34213	34213	33985	33621

Table: The number of coincidence classes found by different lemmas

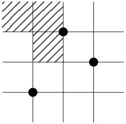
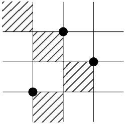
Generalized method

After fixing a force F, we can recursively apply a variant of the previous lemma to capture even more coincidences.

- Walk down a decision tree, branching on
  - box is empty,
  - box contains at least one point
- Leaves should give a contradiction, or an instance of the target pattern

Example

- Prove that containment of p implies containment of p'

$p =$  $p' =$ 

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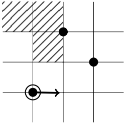
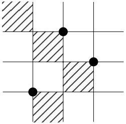
$p =$  $p' =$ 

- Use force $F = ((1, E))$

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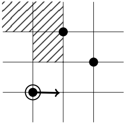
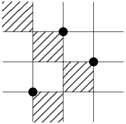
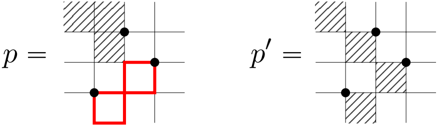
$p =$  $p' =$ 

- Use force $F = ((1, E))$
- Consider a permutation π which contains p, and take the occurrence of p that has maximum strength w.r.t. F

Example

- Want to add shading to $\ulcorner 1, 0 \urcorner$ and $\ulcorner 2, 1 \urcorner$



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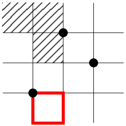
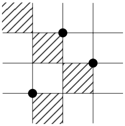
$p =$  $p' =$ 

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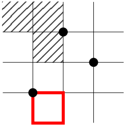
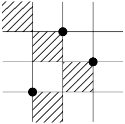
$p =$  $p' =$ 

- Consider $\ulcorner 1, 0 \urcorner$
  - If empty, we're done

Example

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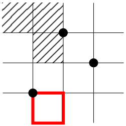
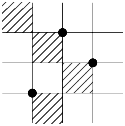
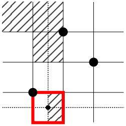
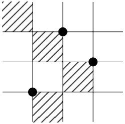
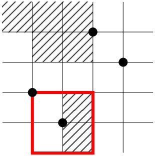
$p =$  $p' =$ 

- Consider $\lceil\!\lfloor 1, 0 \rfloor\!\rceil$
  - If empty, we're done
  - Otherwise, consider the rightmost point

Example

$p =$  $p' =$ 

$w_1 = p^{\overrightarrow{\ulcorner 1, 0 \urcorner}} =$ 

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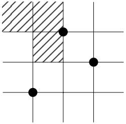
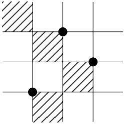
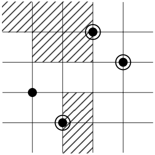
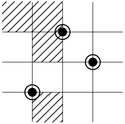
$p =$ 

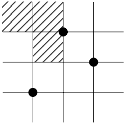
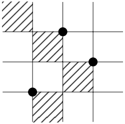
$p' =$ 

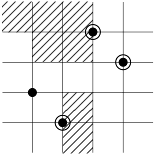
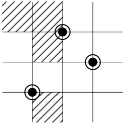
$w_1 = p^{\overrightarrow{\ulcorner 1, 0 \urcorner}} =$ 

$q_1 =$ 

- Consider subsequence at indices 234

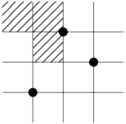
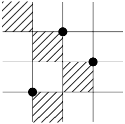
Example

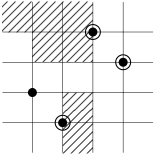
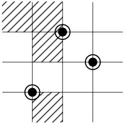
$p =$  $p' =$ 

$w_1 = p^{\overrightarrow{\ulcorner 1, 0 \urcorner}} =$  $q_1 =$ 

- Consider subsequence at indices 234
- q_1 is an occurrence of p that is stronger, a contradiction

Example

$p =$  $p' =$ 

$w_1 = p^{\overrightarrow{\ulcorner 1, 0 \urcorner}} =$  $q_1 =$ 

- Consider subsequence at indices 234
- q_1 is an occurrence of p that is stronger, a contradiction
- Can assume that $\ulcorner 1, 0 \urcorner$ is empty

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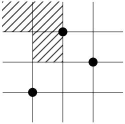
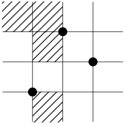
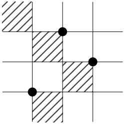
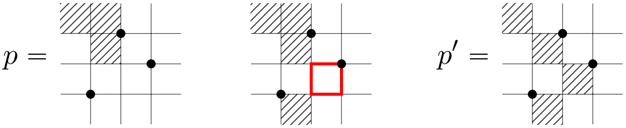
$p =$ 



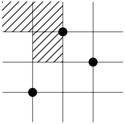
$p' =$ 

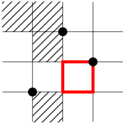
Example

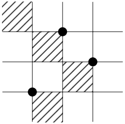


- Now consider $[2, 1]$

Example

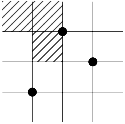
$p =$ 

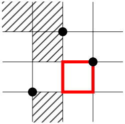


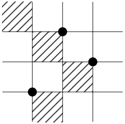
$p' =$ 

- Now consider $[2, 1]$
  - If empty, we're done

Example

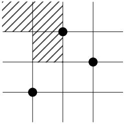
$p =$ 

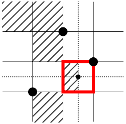


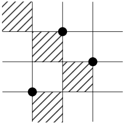
$p' =$ 

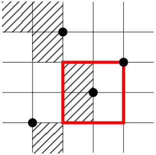
- Now consider $\ulcorner 2, 1 \urcorner$
  - If empty, we're done
  - Otherwise, consider the leftmost point

Example

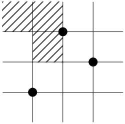
$p =$ 

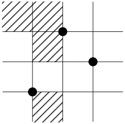


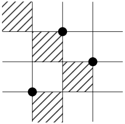
$p' =$ 

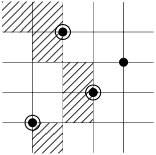
$w_2 =$ 

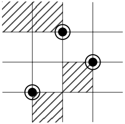
Example

$p =$ 



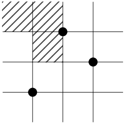
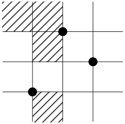
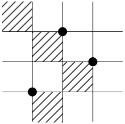
$p' =$ 

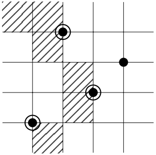
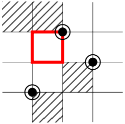
$w_2 =$ 

$q_2 =$ 

- Consider subsequence at indices 123

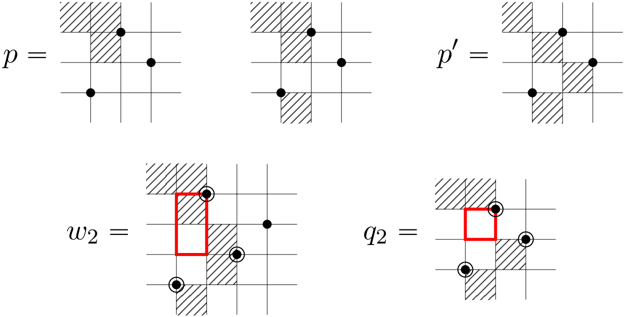
Example

$p =$   $p' =$ 

$w_2 =$  $q_2 =$ 

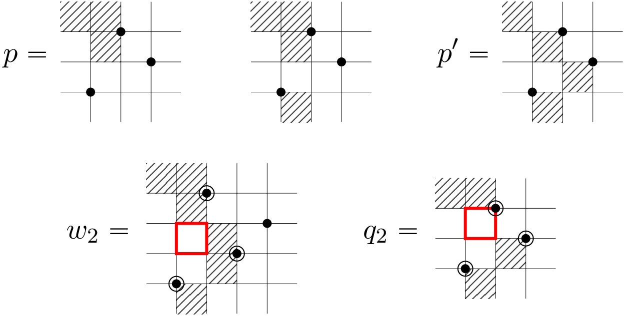
- Consider subsequence at indices 123
- Looking at q_2, we're missing $\ulcorner 1, 2 \urcorner$

Example



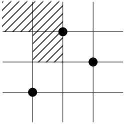
- Consider subsequence at indices 123
- Looking at q_2, we're missing $[1,2]$
- Corresponds to $[1,2]$ and $[1,3]$ in w_2

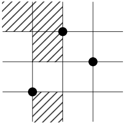
Example

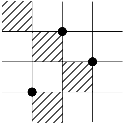


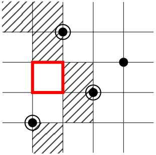
- Consider subsequence at indices 123
- Looking at q_2, we're missing $[1, 2]$
- Corresponds to $[1, 2]$ in w_2

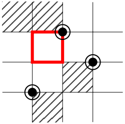
Example

$p =$ 



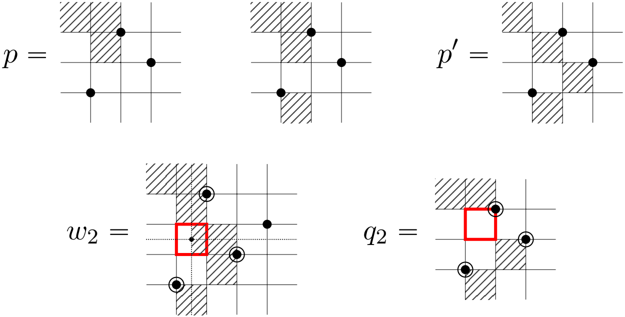
$p' =$ 

$w_2 =$ 

$q_2 =$ 

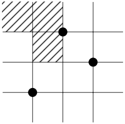
- Consider subsequence at indices 123
- Looking at q_2, we're missing $\llcorner\!\ulcorner 1, 2 \urcorner\!\lrcorner$
- Corresponds to $\llcorner\!\ulcorner 1, 2 \urcorner\!\lrcorner$ in w_2
    - If empty, we're done

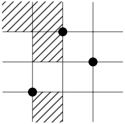
Example

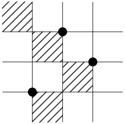


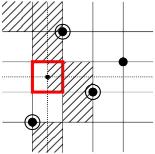
- Consider subsequence at indices 123
- Looking at q_2, we're missing $[1, 2]$
- Corresponds to $[1, 2]$ in w_2
  - If empty, we're done
  - Otherwise, consider the rightmost point

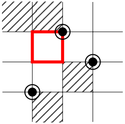
Example

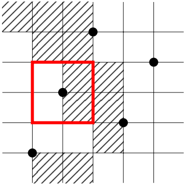
$p =$ 



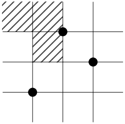
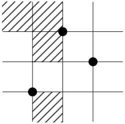
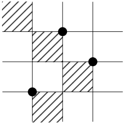
$p' =$ 

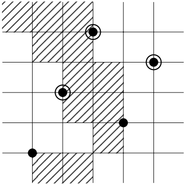
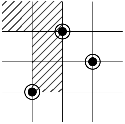
$w_2 =$ 

$q_2 =$ 

$w_3 =$ 

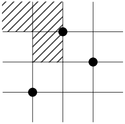
Example

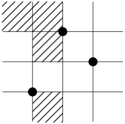
$p =$   $p' =$ 

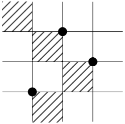
$w_3 =$  $q_3 =$ 

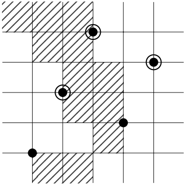
- Consider subsequence at indices 235

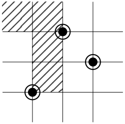
Example

$p =$ 



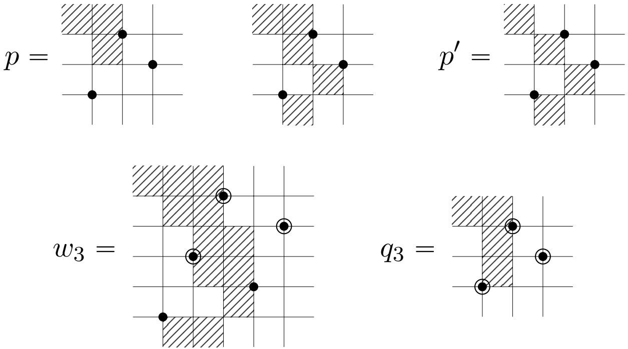
$p' =$ 

$w_3 =$ 

$q_3 =$ 

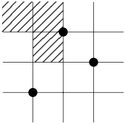
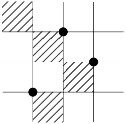
- Consider subsequence at indices 235
- q_3 is an occurrence of p that is stronger, a contradiction

Example



- Consider subsequence at indices 235
- q_3 is an occurrence of p that is stronger, a contradiction
- Can assume that $\ulcorner 2, 1 \lrcorner$ is empty as well

Example

$p =$  $p' =$ 

- We've proved that the occurrence with maximum strength w.r.t. F has no points in $\ulcorner 1, 0 \lrcorner$, $\ulcorner 2, 1 \lrcorner$
- Hence this is an occurrence of p' as well $\square$

Algorithm

We have formalized this into an algorithm (see abstract) and implemented it in Python. It is available on GitHub:

`http://tinyurl.com/shadingalgorithm`

- Currently not very user friendly
- Exhaustively search for proofs that recurse no more than d steps
- Has some optimizations, like deriving a force retroactively

Results

Let d denote the maximum recursion depth used in the proof.

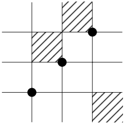
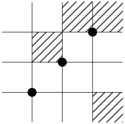
Pattern	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
123	33634	33602	33548	33540	33538	33536	?
132	33621	33459	33412	33395	33394	33394	33390

Table: The number of coincidence classes found by different depths

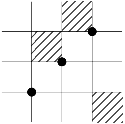
Full coincidence classification of length 3 mesh patterns

After dealing with a handful of special cases we show that the number of coincidence classes for the patterns in Table 2 are 33516 and 33350, respectively.

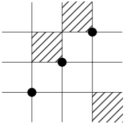
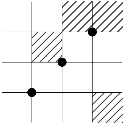
One of those special cases

$p =$  $p' =$ 

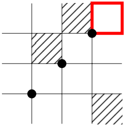
- Let force $F = ((1,S),(2,W),(3,W))$



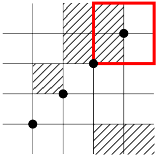
One of those special cases

$p =$  $p' =$ 

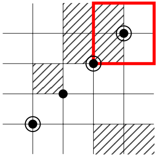
- Let force $F = ((1, S), (2, W), (3, W))$



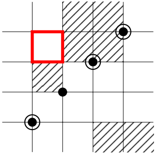
One of those special cases



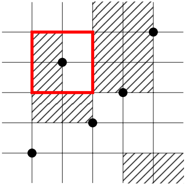
One of those special cases



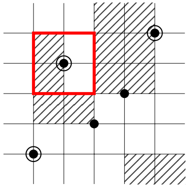
One of those special cases



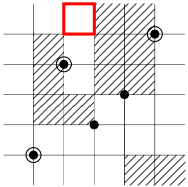
One of those special cases



One of those special cases



One of those special cases



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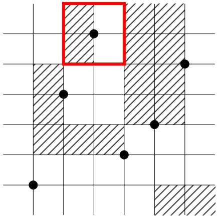
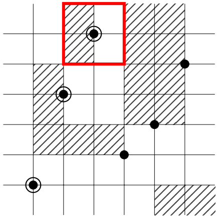
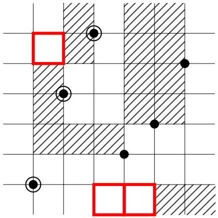


One of those special cases



One of those special cases



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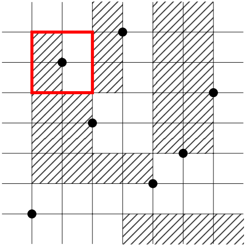
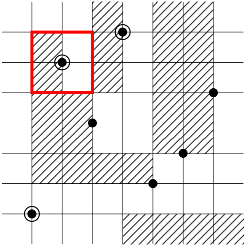
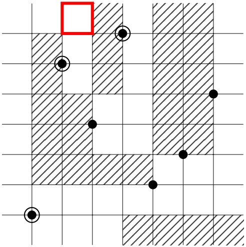


One of those special cases



One of those special cases



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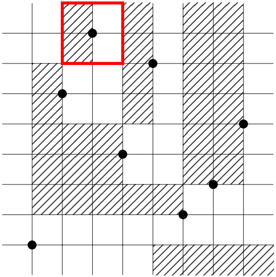

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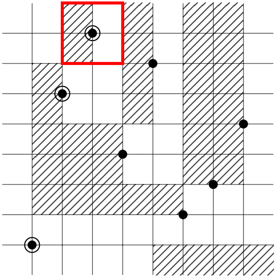

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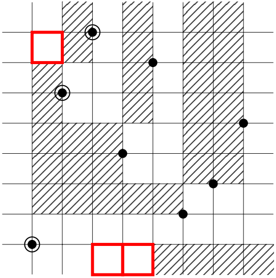

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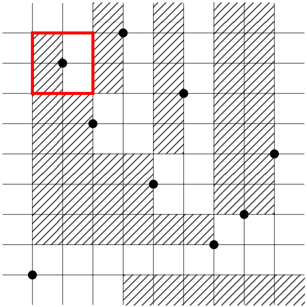

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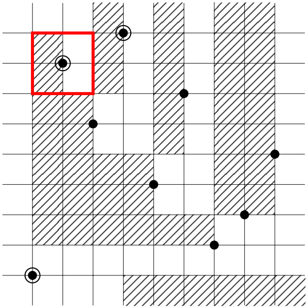

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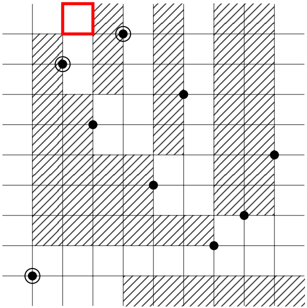

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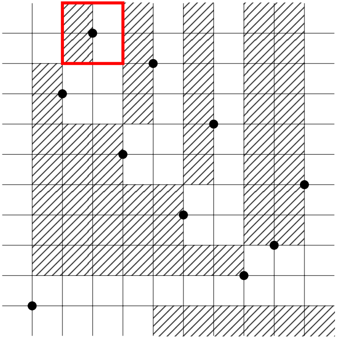

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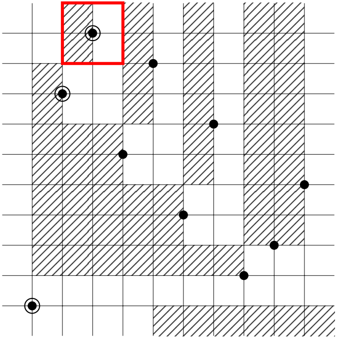

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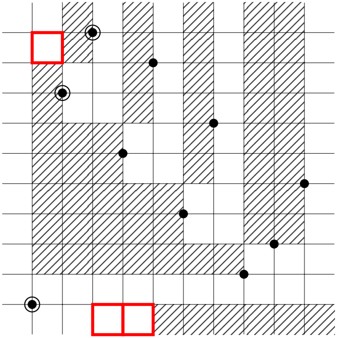

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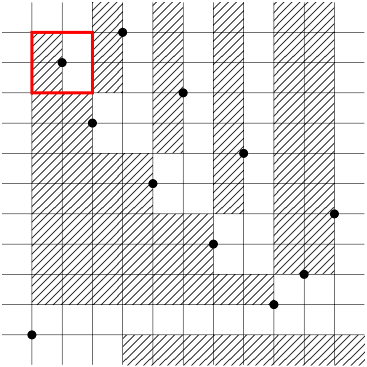

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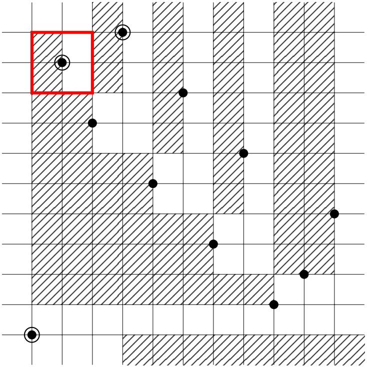

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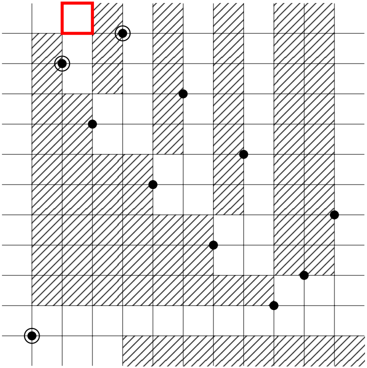

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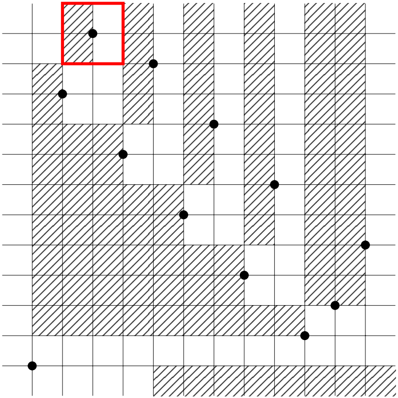

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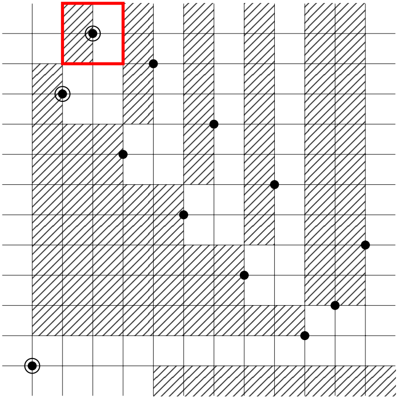

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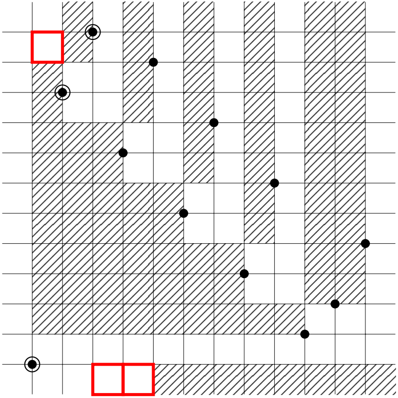

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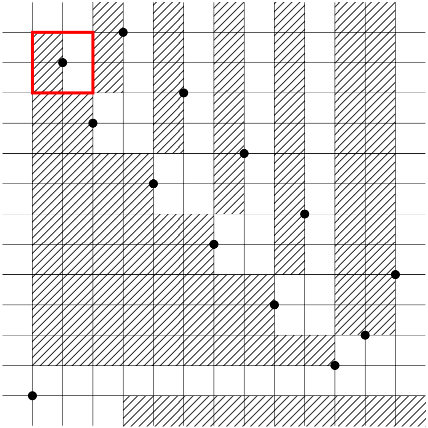

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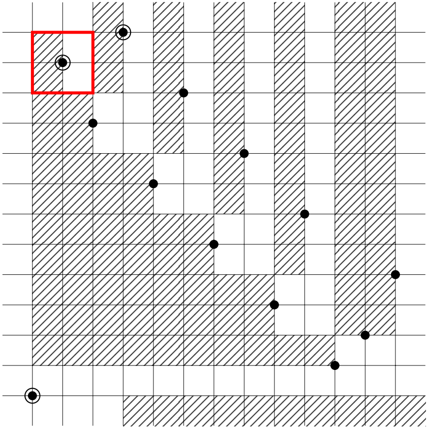

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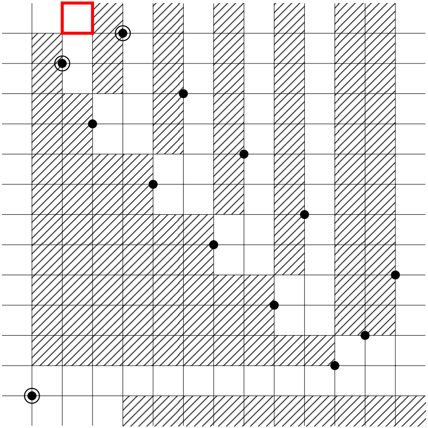

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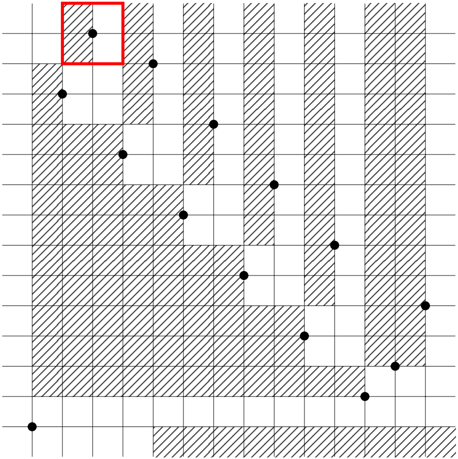

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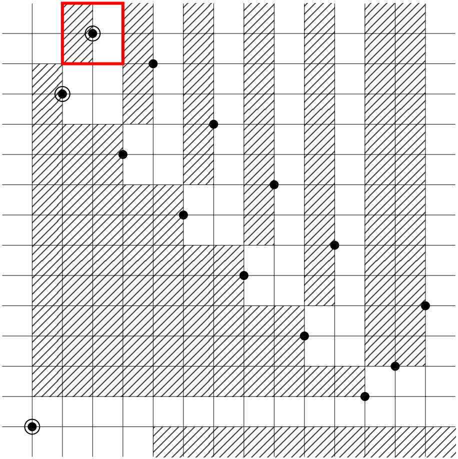

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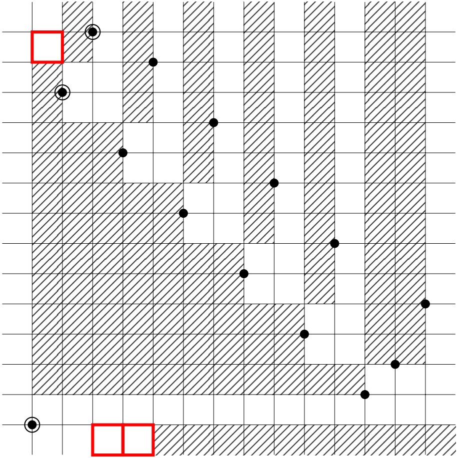

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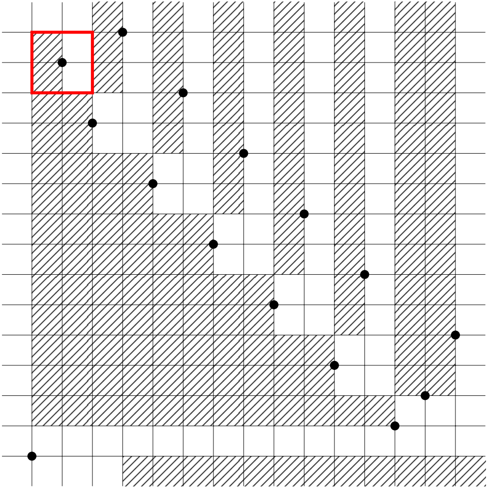

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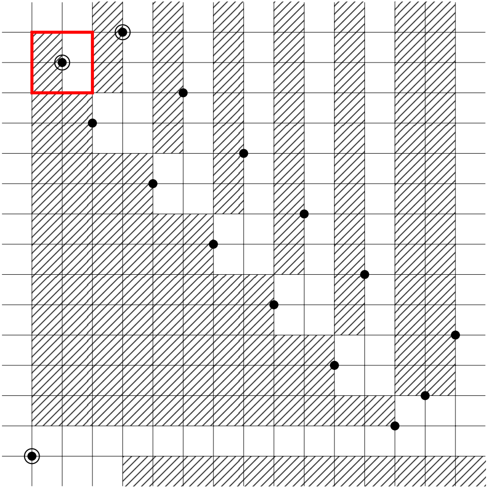

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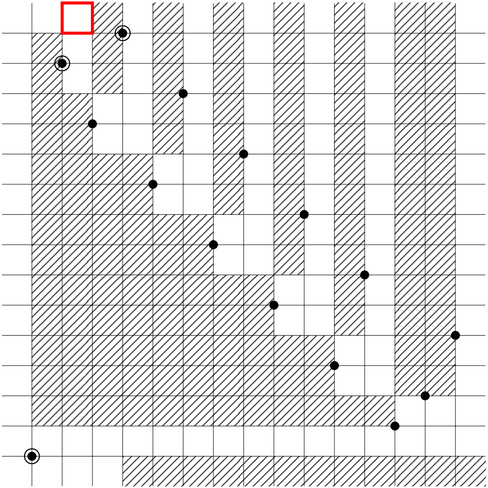

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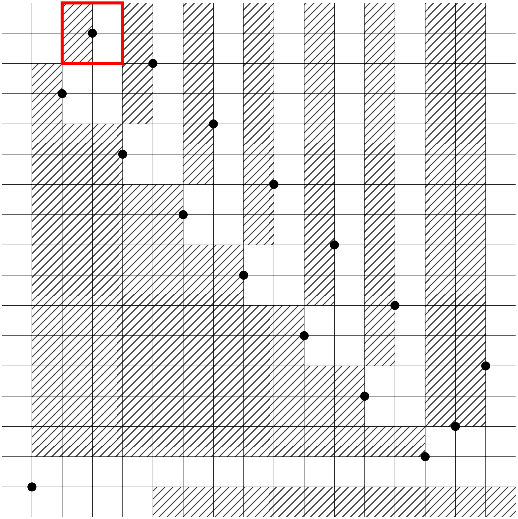

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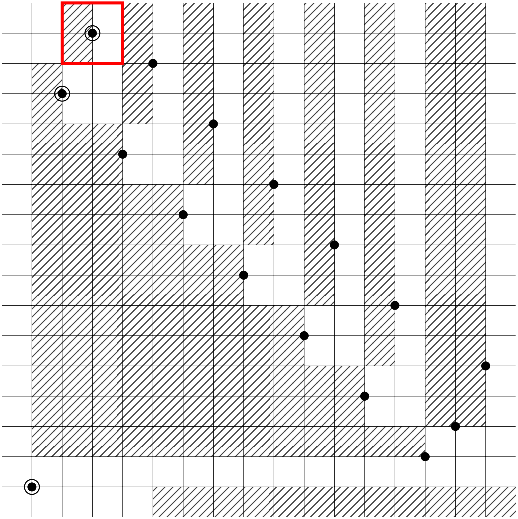

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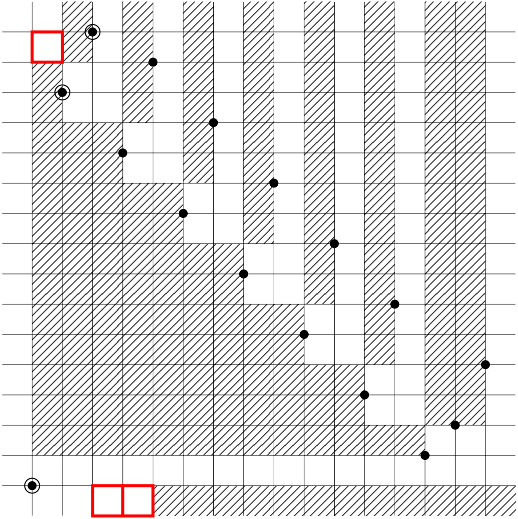

Example

For a finite permutation, this process will eventually exhaust all points. When that happens, we must have found a new occurrence that is stronger w.r.t. F, a contradiction.

Future work

- Find a way to handle “infinite” cases neatly
- Support more complex conditions than just a point in a box, e.g.
    - point in either of two boxes,
    - inversion across two consecutive boxes
- Use unidirectional containment to prove coincidences
- Classify length 4 patterns

Thanks for listening!

Claesson, Tenner, and Ulfarsson.
Coincidence among families of mesh patterns.
Australas. J. Combin., 2015.

Hilmarsson, Jónsdóttir, Sigurðardóttir, Viðarsdóttir, and Ulfarsson.
Wilf-classification of mesh patterns of short length.
Electron. J. Combin, 2015.